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Higher-Order Quantum Logics

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Until now quantum logics has been first-order, but physics requires higher-order logics. We construct a natural higher-order language Q for quantum physics. Q is a finitistic logic based on Peano set theory and Grassmann algebra. Higher-order predicates are identified with their extensions, higher-rank sets. QAND and QOR (the AND and OR of Q) are naturally noncommutative but reduce to the commutative lattice operations for the first-order part of the language. We form higher-order predicates and sets by a setting operator similar to Peano's ι that forms a simple extensor $\iota\psi = \{\psi\}$ from any extensor ψ. In a note added in proof, we correct Q so that a bond like $\{\{\alpha, \beta\}\}$ between two fermions α and β is a quasiboson, as the application to lattice chromodynamics strongly suggests.

1. INTRODUCTION

Perhaps it is time to attempt once more, as Leibniz did, to set up a universal formal language for physics. Here I propose the core of such a language called Q. Set theory may be regarded as a universal grammar or syntax for classical physics. Q provides set theory with a definite physical interpretation, so that it may be regarded as a language as well as a syntax, and revises it in the light of relativity and quantum theory.

The work of von Neumann suggests that the logical particles of Q, the quantum equivalents of AND, OR, NOT, and IF, should be those of an orthomodular lattice. While this suggestion has been stimulating, in fact the lattice language has been of little use as a way to express new physical theories. The language Q uses a logics that is closer to ordinary physical practice, more general than the von Neumann lattice logics in some respects, and more special in others.

Q is more general than lattice logics in that the Q operations corresponding to AND and OR, which I call QAND and QOR, are noncommutative as well as nondistributive, and nilpotent rather than idempotent, although they

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reduce to orthomodular lattice operations in appropriate special cases. Q is an extensor algebra; not a lattice, but a noncommutative generalization of one; and not a poset, but a generalized poset with two related partial orders.

To be sure, no choice of logical structure can be judged until the rest of the structure is provided, any more than one can judge a statue from the pedestal. A language needs something to go between its particles, like verbs or nouns.

The standard language for mathematics is set theory, which supplements the above logical particles by an infinite set of nouns, the names of sets. Indeed, von Neumann spoke of his discovery not only as quantum logics but also "quantum set theory."

There is urgent physical need for quantum set theory. A topology is a set of sets, and so presumably a quantum topology is a quantum set of quantum sets. Thus a quantum description of spacetime and its contents calls for a quantum theory of second-rank sets, or equivalently, second-order predicates. (Henceforth I use "order" both for sets and predicates.) Q includes a higher-order quantum set theory, which identifies quantum sets with Fermi–Dirac ensembles.

While Q is built on verbs rather than nouns, these have close correspondences to the nouns of the usual set theory.

2. EXTENSOR LOGICS

Von Neumann (1932) arrived at lattice logics by abstraction from the subspaces of Hilbert space, which form the predicate of the empirical logic of quantum mechanics. The von Neumann logic of a quantum system is an algebra of the subspaces of its Hilbert space.

Grassmann (1911) proposed a beautiful linear-algebraic theory of subspaces, called extensor algebra here, which is richer than von Neumann's. Q uses Grassmann's theory of subspaces rather than von Neumann's. Both may start from a quantum system S described in a Hilbert space V with a Hilbert isomorphism $\dagger$ to its dual space $\dagger V$; I write operators to the left of their operands, even the operators $\dagger$ and $\perp$. The alternative to the lattice $L(V)$ of the space V is an extensor algebra $E(V)$ over the same space.

First let me sketch the structure and quantum application of Grassmann's extensor algebra $E = E(V)$. The algebra E is not merely one exterior algebra (in the modern sense) but two, head to tail, with two interwoven products corresponding to the two lattice operations.

The fundamental elements of Grassmann's theory are called extensors (*Ausdehnungen*) and are elements of E. They are the simpler nontrivial words of the language Q and represent coherent uniform input operations (that is, input from experimenter to experimentee), which I call input vectors and write as $\langle \alpha|$ or ψ or (ψ'').

Let N be the dimension of the initial linear space V. Then E has a fundamental skew-symmetric N-ic form $[\psi_1 \dots \psi_N]$ defining an invariant volume, called the Grassmann's form. I omit the technicalities of the extension to $N = \aleph_0$, which resembles Dirac's hole theory, and leads to a topology for the space of predicates quite different from Hilbert space.

Dual extensors (never needed by Grassmann, I believe, but indispensable for physics) represent outputs and are written $|\beta\langle = \phi = (\phi_n)$. The value or contraction of a dual extensor ϕ with an extensor ψ is written $|\beta\langle\alpha| = \phi(\psi) = \phi_n \psi^n$. The algebra $E(V)$ is a Hilbert space like V, with addition $+$ and multiplication by field elements having the usual quantum meanings. I envisage replacing the complex coefficients by integers in a later, more fundamental theory.

Like a lattice, an extensor algebra E has two associative products, designated by $\vee$ (which will be our QOR, the quantum OR) and $\wedge$ (our QAND, the quantum AND). Grassmann named these the progressive and regressive products. Peirce's principle (that the inventor has the right to name the invention) forbids changing these terms. The Grassmann product usually designated by $\wedge$ in contemporary exterior algebra is actually $\vee$.

The extensor algebra $E(V)$ is graded, with grade, called degree (*Stufe*) by Grassmann, ranging from 0 to N, the dimension of V. This grade corresponds to the cardinality of classical set theory, to the modulus of lattice theory, and to the multiplicity (degree of degeneracy) of energy levels of ordinary quantum parlance.

Grassmann placed the greatest possible emphasis upon a dual symmetry between $\wedge$ and $\vee$ expressed by a map $\perp: E(V) \to E(V)$ called complementation (*Ergänzung*) by Grassmann and called the Hodge dual nowadays, to the absurd extent of refusing to introduce distinct multiplication signs for the two products, relying instead on context to distinguish them. (The signs $\wedge$ and $\vee$ are due to Peano.) The Grassmann complement is our negation operation, the quantum NOT. It complements degree; if ψ has degree g, then $\perp\psi$ has degree $N-g$. Set theory breaks the Grassmann symmetry: the unit set $\{a\}$ has degree 1, not $N-1$.

I assume also an antilinear degree-preserving anti-isomorphism designated by $\dagger$ from $E(V)$ to its dual space $\dagger E(V)$. This $\dagger$ is the Hilbert $\dagger$ operator of E induced by the Hilbert $\dagger$ operator of V, and interchanges $\wedge$ and $\vee$. The Hilbert norm of an extensor ψ is written $\|\psi\| = \dagger\psi(\psi)$. I mention without detail that in the application to relativistic functional quantum theory, $\dagger$ is not an invariant fixed element of structure, but instead is but one of an infinite class of $\dagger$'s all on the same footing, while the Grassmann form, progressive product, and sum in E are invariant concepts even in the relativistic case.

The Hilbert dual and Grassmann complement operators $\dagger$ and $\perp$ uniquely determine each other. They are the same element of structure expressed in different terms. The Hilbert dual $\dagger\psi$ outputs the same kind of quantum that ψ inputs, while the Grassmann complement $\perp\psi$ inputs every kind of quantum but the ones that ψ inputs.

Extensors have three ascending levels of generality relative to a given basis β for V:

- Extensors which are products of basis vectors of β are called basic.
- Those which are products of vectors are called simple.
- Those which are not simple are called compound; Grassmann called them imaginary.

We designate the ray of extensor ψ by $[\psi]$. If $[\psi]=[\chi]$, we write $\psi\equiv\chi$ and say that ψ is projectively equal to χ. We call the ray $[\psi]$ basic, simple, or compound as we do the vector ψ. The term basic also applies to subspaces of V, those which are spans of basis vectors of the basis β. Extensors ψ and ϕ are called disjoint when $\psi\wedge\varphi=0$.

Now the correspondence to the usual lattice logics can be stated:

Classical Logics

For disjoint basic rays in $E(V)$, the natural operations induced by $\vee$, $\wedge$, and $\perp$ are isomorphic to the Boolean operations $\cup$, $\cap$, and $\neg$ on the subsets of β. This is c logic.

Classical-Quantum Logics

For disjoint simple rays the same operations agree with those of an orthomodular complemented lattice, the usual lattice of subspaces of V. In particular, simple extensors commute projectively: $\psi\vee\chi\equiv\chi\vee\psi$. This is cq logic.

Quantum Logics

For disjoint general rays these operations do not agree with those of a lattice, since compound extensors need not commute projectively. This is q logic.

Thus extensor logics is a proper noncommutative generalization of the von Neumann lattice logics, which is a proper nondistributive generalization of Boolean algebra.

3. INTERPRETATION

To understand and justify this extension of quantum logics, it is helpful to recall a certain systematic difference between the way Heisenberg and Bohr interpret quantum input–output (io) vectors and the way that Born and von Neumann do.

For background, we first recall a similar difference between two concepts of a heat reservoir R in statistical thermodynamics, the individual and the statistical:

- We can think of R as an arbitrary large body in thermal equilibrium with the individual system S.
- Or we can represent R as a standard heat reservoir, namely a large ensemble of systems identical to S.

The two interpretations, individual and statistical, respectively, are expected to be physically equivalent. Schrödinger (in his statistical thermodynamics) adopts the statistical representation of the heat reservoir because it is definite, uniform, and simple.

Similarly, we can think of an input vector ϕ in two physically equivalent ways, individual (the Bohr–Heisenberg interpretation) and statistical (the Born–von Neumann interpretation). Either:

- Each vector ϕ represents an equivalence class of ways of preparing the individual quantum system S, as in the classic operational interpretations of Ludwig and of Foulis and Randall, for example.
- Or the vector ϕ represents one standard way of preparing the individual quantum system; namely, it represents a large ensemble of systems isomorphic in structure to S, from which we select S at random. Briefly, ϕ describes a set of S's.

Again the statistical representation of an input operation is simpler and more definite than the individual interpretation, and we adopt it. But there are more sets of quanta than von Neumann envisaged.

Since a set of S's is exactly a Fermi–Dirac ensemble of S's, and Fermi–Dirac ensembles are described by multivectors (skew tensors), we may specify a set of S's by a multivector. In the presence of the Hilbert inner product, which defines the complement operation, a multivector is also an extensor.

It is simple to imbed the von Neumann lattice logics within the extensor logics. Each element of the von Neumann lattice logics is a subspace, and may be represented by the simple extensor formed by $\vee$-multiplying the vectors in a basis for the subspace. The lattice order $[\phi] \subset [\psi]$ for the rays of simple extensors ϕ and ψ holds, by definition, when for some extensor χ, $\psi = \chi \vee \phi$.

In addition there are compound extensors. Grassmann had no meaning for these precisely because they do not represent subspaces. They represent quantum superpositions of preparations described by subspaces, and represent preparations of S not envisaged by von Neumann.

For example, extensors of degree 1 represent "pure states" in the von Neumann logics, while those of degree 2 represent "mixed states," mixtures of two degree-1 inputs. The idea of a coherent superposition of a first-degree input with a second-degree one, or of two second-degree inputs, does not occur in the von Neumann quantum logics, but is routine in extensor logics.

An interpretation of an input vector is an experimental input operation described by the vector. It is not difficult to design physical input operations to go with these theoretical ones. We may carry out such an operation in two steps: We form a coherent superposition of ensemble vectors, and then extract a member of the resulting ensemble.

- To form a coherent superposition of a one-fermion and two-fermion input, to be sure, violates the Wick–Wightman–Wigner statistics superselection law, and may present some extra difficulty. But nothing prevents the formation of coherent superpositions of two degree-2 input vectors; for example, a di-fermion of spin $S=1$ may be resolved by a Stern–Gerlach operation into a coherent superposition of vectors with z component of spin $S_z=1$, 0, and -1.
- The operation of extracting one member of an ensemble at random may be approximated by a stripping reaction, where a projectile combines with one particle of a target and carries it off. We must carry out the stripping operation on the contents of the target region without determining the contents more precisely than by the prior preparation.

The duals to these input operations are output operations. I omit their physical description here.

The von Neumann logics could describe this input operation by a statistical operator in the single-fermion Hilbert space. This would lose phase information that is retained in the extensor description. The extensor logics is a proper extension of the von Neumann quantum logics in that its single-fermion experiments are not described by the usual quantum theory of the single fermion. (They may, to be sure, be described within the von Neumann logics of the many-fermion ensemble. However, the extensor logics of the many-fermion ensemble is richer still.)

There is also a gain in simplicity. The von Neumann logics describe predicates and sets of electrons by different algebras, lattices and exterior algebras, respectively. This is alien to classical logics, where (hereditarily finite) predicates and sets are isomorphic, the sets being the extensions of the predicates. Here we represent both quantum predicates and quantum sets by the same extensor algebras, restoring extensionality.

4. HIGHER-ORDER QUANTUM LOGICS

So much for the first-order predicate algebra. We now discuss the higher-order quantum predicate algebra. This theory is still somewhat speculative, and is directed toward theories of quantum spacetime that are not yet connected with experiment; but the formal indications for the higher-order theory is so strong that I am sure it is along the right track.

Classical Construction

In general if ψ is regarded as a predicate, then $\{\psi\}$ is a predicate of predicates, holding only for the predicate ψ. If ψ is interpreted as a set, then $\{\psi\}$ represents a set of sets and its sole element is ψ.

For simplicity I limit attention to pure set theory, with no proper elements. There is only one first-order predicate 1, the null predicate. In classical logics we apply a bracket to the first-order predicate 1 to construct a second-order predicate $\{1\}$; then $\{1\}$ represents the property of being the predicate 1. If the predicates of order $\leq N$ make up the Boolean algebra B^N, then their brackets generate a Boolean algebra B^{N+1} made up of predicates of order $\leq N+1$.

Following Peano, I designate $\{\psi\}$ by $\iota\psi$; we used $Q\psi$ for $\iota\psi$ in Finkelstein *et al.* (1979) before we knew of Peano's work (Q standing for "quantized" or "quantified," because $Q\psi$ belongs to the "second quantized" or quantified theory).

If $\psi, \ldots, \chi$ are all distinct predicates of order $\leq N$, then

$$\{\psi, \ldots, \chi\} = \iota\psi \vee \cdots \vee \iota\chi \tag{1}$$

has order $\leq N+1$. More generally, we form the class B^{N+1} of all predicates of order $\leq N+1$ by applying ι to all the elements of B^N [calling the set of all these ι-images $\iota'(B^N)$] and then closing $\iota'(B^N)$ under the operations $\wedge$, $\vee$, and $\perp$:

$$B^{N+1}(S) = \text{closure}(\iota'(B^N)$$

This process is iterated to form sets of any finite order R, involving up to R nesting brackets or consecutive ι's. These are all described in one set theory S, built from the null set 1. The elements of S are generated from 1 by finite numbers of the operations $\vee$, $\wedge$, $\perp$, and ι, subject to familiar identities.

Quantum Construction

The beauty of the extensor logics is that it provides these classical procedures with close parallels in the quantum theory. Heuristically speaking, we do to basis vectors of a vector space V what is classically done to points of a possibility space or phase space S. For example, the extensor algebra $E(V)$ is the quantum correspondent of the Boolean algebra $B(S)$.

The quantum ι applied to the unit extensor 1 (which represents the null set or predicate) gives a new second-order extensor $\iota 1$. In general, from the extensors of order N we form those of order $N+1$ by bracketing (applying ι) and closing under the extensor operations $\wedge$, $\vee$, $\perp$, 1. Call this process E. By iterating we form sets of all orders $E^R(1)$; the least value of R for which $E^R(1)$ contains an extensor ψ is called the order of ψ. The union of the $E^R(1)$ is an infinite-dimensional extensor algebra $E^\infty(1)$. The basic extensors of $E^\infty(1)$ are generated from 1 by finite numbers of the operations $\vee$, $\wedge$, $\neg$, and ι subject to familiar identities. Every basic extensor of $E^\infty(1)$ is either of finite degree or finite codegree.

In the quantum theory we form a Hilbert space Q from the extensor algebra $E^\infty(1)$ in a routine way. There is a natural Hilbert norm on $E^\infty(1)$. Closure of $E^\infty(1)$ with respect to that norm yields the Hilbert space Q.

The operator ι raises order and its Hermitian adjoint

$$^\dagger\iota := \dagger\iota\dagger \tag{2}$$

lowers order. We may normalize ι so that it obeys the familiar Bose–Einstein relation

$$^\dagger\iota\iota - \iota^\dagger\iota = 1$$

which also holds between the differential operators d/dx and x. This does not affect the action of ι on basic rays, which must agree with classical set theory.

For sets of the special form $\iota^n 1$, which Peano identified with the integers, the operator

$$R := \iota\dagger\iota \tag{3}$$

is precisely the order operator. In general, an eigenvector of R with eigenvalue r has the form $\iota^r\alpha$, where α is not of the form $\iota\beta$. We call the operator R the rank.

The physical need for the bracket and ι arises in quantum mechanics and field quantum field theory when we must couple dynamical variables with spacetime points to define trajectories or fields. As long as spacetime is a classical set it is possible and customary to use the classical bracket to couple variables with coordinates. If spacetime is a quantum set (which is a Fermi–Dirac ensemble, we recall) then we might use the quantum bracket for this purpose, to do quantum field theory on quantum spacetime.

I do not advocate this use of the bracket, which copies classical field theories too literally. The main field theory is gravity, which describes the causal connection. In the quantum case, it is more natural to use the bracket to couple spacetime points directly to each other, and so to describe the causal connection by a network rather than a field. With the network as dynamical variable there is no fundamental need for fields. This approach was attempted in Finkelstein (1989) and is still under study. It is the main motivation for Q.

5. QUANTIFIERS

Physicists have dealt with quantification in quantum logics elegantly since quite early in the development of quantum theory. If $\psi \in V$ describes one fermion, so that an ensemble is described by an extensor $\Psi \in E(V)$, then in physics one uses the numerical quantifier $N(\psi)$ (for "the number of fermions of the kind ψ," also called the occupation number operator) in preference to the Aristotelean quantifiers $\forall$ ("for every fermion") and $\exists$ ("for some fermion"). $N(\psi)$ is a linear operator on $E(V) \to E(V)$ defined in Q as follows.

Let us write $\psi^\vee$ for the linear operator on extensors of left $\vee$-multiplication by ψ; for all $\psi, \chi \in E(V)$,

$$\psi^\vee \chi := \psi \vee \chi \in E(V) \tag{4}$$

This $\psi^\vee$ is a creation operator. Similarly we write $\psi^\wedge$ for left $\wedge$-multiplication by ψ. We would designate the corresponding right multiplications by ${}^\vee\psi$ and ${}^\wedge\psi$.

The Hermitian adjoint $\dagger\psi^\vee\dagger =: {}^\dagger(\psi^\vee)$ is an annihilation operator, and is identical with $(\perp\psi)^\wedge$, left $\wedge$-multiplication by $\perp\psi$. (I thank G.-C. Rota for pointing this out.)

Let us normalize ψ to 1. Then ψ and ${}^\dagger\psi$ obey "canonical anticommutation relations" and one defines the number operator $N(\psi)$ by

$$N(\psi) := \psi^\dagger\psi \tag{5}$$

For every normalized ψ, $N(\psi)$ is a positive linear operator mapping $E(V) \to E(V)$ with eigenvalues $N' = 0, 1$.

The eigenvectors of $N(\psi)$ belonging to eigenvalue $N' = 0, 1$ are the homogeneous extensors of degree N' in ψ. Those with $N' = 1$ are of the form $\psi \vee \alpha$ for some α; those with $N' = 0$ lack any factor of ψ and are annihilated by ${}^\dagger\psi$. Thus $N(\psi)$ agrees with the classical notion of the number of ψ's.

In some applications we may dispense with variables in our predicate logic. Free variables are metalinguistic conveniences, since expressions with free variables have no meaning in the object language. If bound variables stand for the system under study, and there is only one of these, then its symbol may be taken as read, and we may write $\forall P$ instead of $\forall x P(x)$, and $\exists P$ instead of $\exists x P(x)$. Often in this context $\exists P$ is written $\bigcup P$ and $\forall P$ is written $\bigcap P$.

The classical (finite) existential operator $\bigcup : S \rightarrow S$ is defined by the condition that

$$\bigcup [\{\psi\}] = \psi, \qquad \bigcup [\psi \cup \chi] = \bigcup \psi \cup \bigcup \chi \tag{6}$$

Similarly, we might wish to define the extensor existential operator $\bigvee : Q \rightarrow Q$ by the condition that it be linear and for simple extensors obey

$$\bigvee [\{\psi\}] = \psi, \qquad \bigvee [\psi \vee \chi] = \bigvee \psi \vee \bigvee \chi \tag{7}$$

If either ψ or χ has even degree, however, this leads to the unexpected result

$$\begin{aligned} \bigvee [\{\psi\} \vee \{x\}] &= \psi \vee \chi = \chi \vee \psi = \bigvee [\{\chi\} \vee \{\psi\}] \\ &= -\bigvee [\{\psi\} \vee \{\chi\}] = 0 \end{aligned} \tag{8}$$

This result stems from the difference between the statistics of ψ (quasi-Bose if ψ is of even degree) and $\iota\psi$ (Fermi in every case), or the fact that ι is a superoperator (mixes statistics). What corresponds better than the input extensor ψ to a classical unit predicate $\{\psi\}$ in this respect is the extensor $\{\psi\} \vee \{\dagger\psi\} = P(\psi)$. Because $P(\psi)$ has even degree,

$$\begin{aligned} \bigvee [P(\psi) \vee P(\chi)] &= \psi \vee \dagger\psi \vee \chi \vee \dagger\chi = \chi \vee \dagger\chi \vee \psi \vee \dagger\psi \\ &= \bigvee [P(\chi) \vee P(\psi)] \end{aligned} \tag{9}$$

The numerical quantifier $N(\psi)$ is also an existential quantifier $\bigvee$ in set theory, where N has the spectrum 0, 1. The existential operator is a kind of inverse to ι.

The existential and universal quantifier are related by

$$\bigwedge = \perp \bigvee \perp \tag{10}$$

These may be the best that one can do in Q.

6. FUNCTIONS

While the language of set theory generates its infinite family of nouns with ease and elegance, it is deficient in verbs. It has a small supply of constant verbs ("=," "$\in$,". . .) and no variable ones. The simplest mapping, function, or arrow, the ordered pair $(\alpha \rightarrow \beta)$, representing an operation that transforms α into β, creates insuperable problems for set theory. The usual expression for $(\alpha \rightarrow \beta)$ is the set $\{\alpha\} \cup \{\{\alpha\} \cup \{\beta\}\}$. This choice is lame and gratuitous; it could just as well represent the arrow $(\beta \rightarrow \alpha)$, for example. (For all I know it does; I have not looked up the standard convention, since we will not use it.) Functions act but sets do nothing, and certainly do not act on other sets; they simply *are*.

This was recognized by von Neumann, for example, when he provided his variant of set theory. He took the function concept as primitive and defined sets in terms of functions, instead of attempting the converse. His system was too awkward to be practical, however; largely because it preceded the practice of quantum physics, which provides a working model of how operations should be expressed in a formal language.

In Q the Hilbert dual $\dagger\psi$ is the elemental partial map $\psi \to \bullet$. It is then routine to express general mappings, partial mappings, and multivalued mappings as elements of the enlarged extensor algebra $Q(\dagger)$ formed from 1 by iterating the operations $\wedge$, $\vee$, $\perp$, $\dagger$, and linear combination.

Takeuti (1981) has also put forward a quantum set theory. The two QSTs are supplementary and may be amalgamated. Takeuti's QST is a theory of sets of quantum *variables* of a single quantum system, based on an arbitrary prespecified Hilbert space, and does not identify sets with Fermi–Dirac ensembles; while Q is a theory of sets of quantum *systems*, generates a new Hilbert space, and identifies sets with Fermi–Dirac ensembles. I benefited greatly from Takeuti's work. More recently, the insights of Barnabei *et al.* (1985) on the thought of Grassmann and Peano have been influential.

NOTE ADDED IN PROOF

In this work and in D. Finkelstein and W. Hallidy, Q: A language for quantum-spacetime topology, *International Journal of Theoretical Physics*, **30**, 1991 we identified cardinality (the number of prime factors in a $\vee$-product) with grade (in the sense of graded algebra), assuming that all prime factors had grade 1. Then ι violates grade. This made it impossible to form quasibosons. Even if α and β are fermions of grade 1, so that $\alpha \vee \beta$ has grade 2, the set $\gamma = \iota(\alpha \vee \beta)$ again had cardinality 1 and therefore grade 1. But we wish to use Q in q topology and q network gauge theory. There γ is a topological link between α and β, and should be a quasiboson, of even grade, not a fermion. We have now found a more natural form of Q where γ is a quasiboson. Now the setting-operator ι preserves grade. This also eliminates the paradox of equation (8). I am grateful to J. M. Gibbs, M. Kolodner, W. J. Mantke, and F. (T.) Smith for discussions.

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